

# DEPENDENCE RESULT OF THE WEAK SOLUTION OF ROBIN BOUNDARY VALUE PROBLEMS

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**ABSTRACT.** In this article we establish an approximation result involving the Laplacian with Robin boundary conditions. It informs about the weak solution's dependence from the input function on the boundary.

## 1. Introduction

Let  $\Omega$  be a bounded domain with Lipschitz boundary. We consider the problem of the Laplacian with Robin boundary conditions,

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad (1.1)$$

where  $\nu$  is the outwar normal verctor and  $\beta$  is a measurable positive bounded function on the boundary  $\partial\Omega$ . This kind of problems was extensively studied by many autors, we refer to [1], [2], [3], [7], [5] and references therein for more details.

The aim of this article is to show a dependance result of a sequence of weak solutions  $(u_n)_{n \geq 0}$  with a sequence of input functions  $(\beta)_{n \geq 0}$ . The proof is based on a technical Lemma due to Stampaccia [6].

## 2. Preliminaries and main result

We assume that  $\Omega \subset \mathbb{R}^d$  ( $d \geq 3$ ) is a bounded domain with Lipschitz boundary. We denote by  $\sigma$  the restriction to  $\partial\Omega$  of the  $(d-1)$ -dimensional Hausdorff measure.

We know that the following continous embedding holds,

$$H^1(\Omega) \rightarrow L^q(\Omega), \quad q = \frac{2d}{d-2} \quad (2.1)$$

Moreover each function  $u \in H^1(\Omega)$  has a trace which is in  $L^s(\partial\Omega)$ , where  $s = \frac{2(d-1)}{d-2}$ ; i.e. there is a constant  $c > 0$  such that

$$\|u\|_{s, \partial\Omega} \leq c \|u\|_{H^1(\Omega)} \text{ for all } u \in H^1(\Omega) \quad (2.2)$$

Let  $\lambda > 0$  be a real number,  $f \in L^p(\Omega)$  ( $p > d$ ) and  $\beta$  be a nonnegative bounded measurable function on  $\partial\Omega$ . We consider the following Robin boundary value problem

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$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \beta u = 0 & \text{in } \partial\Omega \end{cases} \quad (2.3)$$

The form associated with the Laplacian with Robin boundary condition is

$$a_\beta(u, v) = \int_{\Omega} \nabla u \nabla v dx + \int_{\partial\Omega} \beta u v d\sigma \text{ for all } u, v \in H^1(\Omega)$$

We start by the definition of the weak solution of the problem (2.3).

**Definition 2.1.** Let  $f \in L^p(\Omega)$ . For each  $\lambda > 0$ , a function  $u = G_\beta^\lambda f \in H^1(\Omega)$  is called a weak solution of the Robin boundary value Problem (associated with  $\beta$ ) if for every  $v \in H^1(\Omega)$

$$a_\beta^\lambda(u, v) = \int_{\Omega} f v dx,$$

where for  $u, v \in H^1(\Omega)$

$$a_\beta^\lambda(u, v) = a_\beta(u, v) + \lambda \int_{\Omega} u v dx$$

It is clear that the closed bilinear form  $a_\beta$  is continuous on  $H^1(\Omega)$  and also coercive on  $H^1(\Omega)$  in the sense that there exists a constant  $c > 0$  such that for all  $u \in H^1(\Omega)$

$$a_\beta^\lambda(u, u) \geq \|u\|_{H^1(\Omega)}^2$$

Let  $L$  be the linear functional on  $H^1(\Omega)$  defined by : for  $v \in H^1(\Omega)$

$$Lv := \int_{\Omega} f v dx$$

Since  $p \geq 2$ , the functional  $L$  is well defined and continuous on  $H^1(\Omega)$ . Thus by coerciveness of the bilinear form  $a_\beta$ , the Lax-Milgram Lemma (see [4, Corollaire V.8 p:84]) implies that there exists a unique weak solution  $u \in H^1(\Omega)$  of the boundary value problem (2.3).

The following lemma is important in the proof of Theorem 2.4, we can find its proof in [6] Lemma 4.1.

**Lemma 2.2.** Let  $\varphi = \varphi(t)$  be a nonnegative, nonincreasing function on the half line  $t \geq k_0 \geq 0$  such that there are positive constants  $c, \alpha$  and  $\delta (\delta > 0)$  such that

$$\varphi(h) \leq c(h - k)^{-\alpha} \varphi(k)^\delta$$

for all  $h > k \geq k_0$ . Then we have

$$\varphi(k_0 + d) = 0, \text{ where } d > 0 \text{ satisfies } d^\alpha = c\varphi(k_0)^{\delta-1} 2^{\delta(\delta-1)}$$

**Theorem 2.3.** Let  $u$  be a weak solution and assume that  $p > d$ . Then

1) if  $\lambda = 0$  and  $\Omega$  is of finite volume, there exists a strictly positive constants  $C_1 = C_1(d, p, |\Omega|)$  such that

$$|u(x)| \leq C_1 \|f\|_p \quad \text{a.e on } \overline{\Omega}$$

2) if  $\lambda > 0$  and  $\Omega$  is an arbitrary domain, there exist a strictly positive constant  $C_2 = C_2(d, p, \lambda)$  such that

$$|G_\beta^\lambda f(x)| \leq C_2 \|f\|_p \quad \text{a.e on } \overline{\Omega}$$

The proof can be found in [7] and is based on the Maza'ya inequality and a standard argument as in Theorem 4.1 of [6].

Our main result is the following Theorem,

**Theorem 2.4.** *Any sequence  $(u_n)_{n \geq 0}$  of weak solutions of the Robin boundary value problem associated to the sequence  $(\beta_n)_{n \geq 0}$  verify the following inequality:*

$$\|u_n - u_m\|_{\infty, \overline{\Omega}} \leq C \|u_n\|_{\infty, \partial\Omega} \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \quad (2.4)$$

for all  $n, m \in \mathbb{N}$  and where  $C$  may depend of  $\lambda$ .

### 3. Proof of Theorem 2.4

*Proof.* Let  $(u_n)_{n \geq 0}$  be a sequence of weak solutions associated with the sequence  $(\beta_n)_{n \geq 0}$ . Let  $k \geq 0$  be a real number and define  $u_{n,m} := u_n - u_m$ .

Define  $v_{n,m} := (|u_{n,m}| - k)^+ \text{sgn}(u_{n,m})$ . Then  $v_{n,m} \in H^1(\Omega)$  and

$$\nabla v_{n,m} = \begin{cases} \nabla u_{n,m} & \text{in } A_{n,m}(k); \\ 0 & \text{otherwise} \end{cases}$$

where  $A_{n,m}(k) = \{x \in \overline{\Omega} : |u_{n,m}(x)| > k\}$ . In the following, we write  $u, v, A(k)$ .. instead of  $u_{n,m}, v_{n,m}, A_{n,m}(k)$ ...

It is clear that  $a_{\beta_n}^\lambda(u_n, v) - a_{\beta_m}^\lambda(u_m, v) = 0$ . Calculating we obtain:

$$\begin{aligned} 0 &= \int_{\Omega} \nabla(u_n - u_m) \nabla v dx + \int_{\partial\Omega} (\beta_n u_n - \beta_m u_m) v d\sigma + \lambda \int_{\Omega} (u_n - u_m) v dx \\ &= \int_{A(k)} |\nabla v|^2 dx + \int_{\partial\Omega} (\beta_n - \beta_m) u_n + \beta_m (u_n - u_m) v d\sigma + \lambda \int_{\Omega} (u_n - u_m) v dx \\ &= \int_{A(k)} |\nabla v|^2 dx + \int_{\partial\Omega \cap A(k)} (\beta_n - \beta_m) u_n v d\sigma + \int_{\partial\Omega \cap A(k)} \beta_m (u_n - u_m) v d\sigma \\ &\quad + \lambda \int_{A(k)} (u_n - u_m) v dx \\ &= \int_{A(k)} |\nabla v|^2 dx + \int_{\partial\Omega \cap A(k)} (\beta_n - \beta_m) u_n v d\sigma + \int_{\partial\Omega \cap A(k)} \beta_m v^2 d\sigma \\ &\quad + k \int_{\partial\Omega \cap A(k)} \beta_m |v| d\sigma + \lambda \int_{A(k)} v^2 dx + \lambda k \int_{A(k)} |v| dx \\ &= a_{\beta_m}^\lambda(v, v) + \int_{\partial\Omega \cap A(k)} (\beta_n - \beta_m) u_n v d\sigma + k \int_{\partial\Omega \cap A(k)} \beta_m |v| d\sigma + \lambda k \int_{A(k)} |v| dx \end{aligned} \quad (3.1)$$

It follows that

$$\begin{aligned} a_{\beta_m}^\lambda(v, v) + \int_{\partial\Omega \cap A(k)} (\beta_n - \beta_m) u_n v d\sigma &= -k \int_{\partial\Omega \cap A(k)} \beta_m |v| d\sigma - \lambda k \int_{A(k)} |v| dx \\ &\leq 0 \end{aligned} \quad (3.2)$$

Which leads to

$$a_{\beta_m}^\lambda(v, v) \leq \int_{\partial\Omega \cap A(k)} (\beta_m - \beta_n) u_n v d\sigma$$

Using the Hölder inequality and (2.2), we obtain the following estimates,

$$\begin{aligned} a_{\beta_m}^\lambda(v, v) &\leq \int_{\partial\Omega \cap A(k)} (\beta_m - \beta_n) u_n v d\sigma \\ &\leq \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \int_{\partial\Omega \cap A(k)} u_n v d\sigma \\ &\leq \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{2, \partial\Omega \cap A(k)} \|v\|_{2, \partial\Omega \cap A(k)} \\ &\leq \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{\frac{1}{2}} |\partial\Omega \cap A(k)|^{\frac{1}{2} - \frac{1}{s}} \|v\|_{s, \partial\Omega} \\ &\leq \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1 - \frac{1}{s}} \|v\|_{s, \partial\Omega} \\ &\leq c \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1 - \frac{1}{s}} \|v\|_{H^1(\Omega)} \end{aligned} \quad (3.3)$$

We have then,

$$\begin{aligned} \alpha \|v\|_{H^1(\Omega)}^2 &\leq a_{\beta_m}^\lambda(v, v) \\ &\leq c \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1 - \frac{1}{s}} \|v\|_{H^1(\Omega)} \end{aligned} \quad (3.4)$$

It follows that

$$\|v\|_{H^1(\Omega)} \leq c_1 \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1 - \frac{1}{s}} \quad (3.5)$$

Using the inequalities (2.1) and (2.2), we obtain the following estimates,

$$\|v\|_{s, \partial\Omega \cap A(k)} \leq c_2 \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1 - \frac{1}{s}} \quad (3.6)$$

and,

$$\|v\|_{q, A(k)} \leq c_3 \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1 - \frac{1}{s}} \quad (3.7)$$

Let now  $h > k \geq 0$ . Then  $A(h) \subset A(k)$  and on  $A(h)$  we have  $|v| \geq h - k$ . It follows that

$$\begin{aligned} \|v\|_{s, \partial\Omega \cap A(k)} &\geq \|v\|_{s, \partial\Omega \cap A(h)} \\ &\geq \| |u| - k \|_{s, \partial\Omega \cap A(h)} \\ &\geq (h - k) |\partial\Omega \cap A(h)|^{\frac{1}{s}} \end{aligned} \quad (3.8)$$

We deduce from (3.6) that

$$(h - k) |\partial\Omega \cap A(h)|^{\frac{1}{s}} \leq c_2 \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1 - \frac{1}{s}}$$

which reduces to,

$$|\partial\Omega \cap A(h)| \leq c_2^s (h - k)^{-s} \|\beta_n - \beta_m\|_{\infty, \partial\Omega}^s \|u_n\|_{\infty, \partial\Omega}^s |\partial\Omega \cap A(k)|^{s-1}$$

Set  $\phi(h) = |\partial\Omega \cap A(h)|$ , we obtain,

$$\phi(h) \leq C(h-k)^{-s}\phi(k)^{s-1}$$

where  $C = c_2^s \|\beta_n - \beta_m\|_{\infty, \partial\Omega}^s \|u_n\|_{\infty, \partial\Omega}^s$ .

As  $s-1 > 1$ , then the conditions of the Lemma 2.2 are satisfied with  $\delta = s-1$  and  $k_0 = 0$ , one obtain  $\phi(d) = 0$  where  $d > 0$  satisfies  $d^s = C\phi(0)^{s-2}2^{(s-1)(s-2)}$ , consequently

$$d = c_4 \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega}$$

and

$$\|u_n - u_m\|_{\infty, \partial\Omega} \leq c_4 \|u_n\|_{\infty, \partial\Omega} \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \quad (3.9)$$

In the same way as in (3.8), we obtain

$$\|v\|_{q, A(k)} \geq (h-k)|A(k)|^{\frac{1}{q}}$$

From (3.7), we deduce

$$(h-k)|A(h)|^{\frac{1}{q}} \leq c_3 \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \|u_n\|_{\infty, \partial\Omega} |\partial\Omega \cap A(k)|^{1-\frac{1}{s}}$$

We take  $k = d$  and  $h = \gamma d$  with  $\gamma > 1$ , we obtain  $|A(\gamma d)| = 0$  which leads to

$$\begin{aligned} \|u_n - u_m\|_{\infty, \Omega} &\leq \gamma d \\ &\leq \gamma c_4 \|u_n\|_{\infty, \partial\Omega} \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \end{aligned} \quad (3.10)$$

From (3.9) and (3.10) we obtain our Theorem.  $\square$

**Corollary 3.1.** *Let  $(u_n)_{n \geq 0}$  be a sequence weak solutions associated with the sequence  $(\beta_n)_{n \geq 0} \in L^\infty(\partial\Omega)$  such that  $\inf_n \beta_n > 0$  then if  $(u_n)_{n \geq 0}$  is uniformly bounded we have for  $p > d$*

$$\|u_n - u_m\|_{\infty, \overline{\Omega}} \leq C \|f\|_p \|\beta_n - \beta_m\|_{\infty, \partial\Omega} \quad (3.11)$$

for all  $n, m \in \mathbb{N}$  and where  $C$  may depend of  $\lambda$ .

In the case where the sequence of weak solutions  $(u_n)_{n \geq 0}$  is uniformly bounded with respect to  $n$  we have the following consequence

**Corollary 3.2.** *Let  $(u_n)_{n \geq 0}$  be a sequence weak solutions associated with the sequence  $(\beta_n)_{n \geq 0} \in L^\infty(\partial\Omega)$  such that  $\inf_n \beta_n > 0$  and  $\lim_n \beta_n(x) = \beta(x)$  a.e  $x \in \partial\Omega$  then if  $(u_n)_{n \geq 0}$  is uniformly bounded we have  $\lim_n u_n(x) = u(x)$  a.e  $x \in \overline{\Omega}$ , where  $u$  is the weak solution associated with  $\beta$ .*

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